Small area estimation in the survey of Lithuanian census

Andrius Čiginas
(joint work with Ieva Burakauskaitė)

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Main objects of the survey

- $\mathcal{U} = \{1, \ldots, N\}$ is a finite census population of individuals.
- There are M domains $\mathcal{U}_1, \ldots, \mathcal{U}_M$ of known sizes $N_1, \ldots, N_M$ such that $\mathcal{U}_1 \cup \cdots \cup \mathcal{U}_M = \mathcal{U}$ and $\mathcal{U}_i \cap \mathcal{U}_j = \emptyset$ as $i \neq j$. For example, the domains are municipalities, $M = 60$.
- Categorical variables of the survey:
  1. *religion* (16 categories);
  2. *mother tongue* (more than 12 categories);
  3. *knowledge of other languages* (16 languages);
  4. *ethnicity* (mass imputation is used).

  It is sufficient to consider binary variables. Let y be one of these with the fixed values $y_1, \ldots, y_N$ in $\mathcal{U}$.
- We aim to estimate the domain proportions

$$\theta_i = \frac{1}{N_i} \sum_{k \in \mathcal{U}_i} y_k, \qquad i = 1, \ldots, M,$$

  or totals $N_i \theta_i$.

Sample design and primary sampling weights

- The sample $s \subset \mathcal{U}$ of size $n < N$ was drawn according to the sampling design $p(\cdot)$ with inclusion into the sample probabilities $\pi_k = \mathrm{P_p}\{k \in s\} > 0$, $k \in \mathcal{U}$.
- We got the sample $s = s^{(1)} \cup s^{(2)} \cup s^{(3)}$, where
  1. the part $s^{(1)}$ contains individuals from the voluntary sample;
  2. $s^{(2)}$ consists of other units which cannot be included into the sampling frame (the part for imputation);
  3. the part $s^{(3)}$ is the probability sample drawn from the sampling frame $\mathcal{U}^{(3)} = \mathcal{U} \backslash \{s^{(1)} \cup s^{(2)}\}$.
- The primary sampling weights are $d_k = 1/\pi_k$, where $\pi_k = 1$ as $k \in s^{(1)} \cup s^{(2)}$, and, in the hth stratum of $\mathcal{U}^{(3)}$,

  $$\pi_k \approx m_k n'_h / N'_h, \qquad k \in s^{(3)},$$

  where N'_h is the stratum size, n'_h is the number of adresses selected, and m_k is the number of persons in the adress.

Regression (calibrated) estimators in domains

- The domain samples $s_i = s \cap \mathcal{U}_i$ are of sizes $n_i \leqslant N_i$.
- Let $\mathbf{x}_k = (1, x_{2k}, \ldots, x_{Pk})'$ be a P-dimensional vector containing the values of auxiliary variables $x_2, \ldots, x_P$ for $k \in \mathcal{U}$, and $\boldsymbol{\theta}_{xi} = \sum_{k \in \mathcal{U}_i} \mathbf{x}_k / N_i$ is the vector of means for each domain $i = 1, \ldots, M$.

The generalized regression estimators (Rao and Molina, 2015)

$$\hat{\theta}_i^{\mathrm{GR}} = \boldsymbol{\theta}'_{xi} \widehat{\mathbf{B}}_i \quad \text{with} \quad \widehat{\mathbf{B}}_i = \left(\sum_{k \in s_i} \frac{\mathbf{x}_k \mathbf{x}'_k}{\pi_k} \right)^{-1} \sum_{k \in s_i} \frac{\mathbf{x}_k y_k}{\pi_k}$$

of θ_i , $i = 1, \ldots, M$, are approximatelly design unbiased if n_i are not small.

The set of variables $x_2, \ldots, x_P$ includes binary variables on age groups, gender, and religions (2011 data) intersected with counties.

The problem

- The direct estimator $\hat{\theta}_i^{\mathrm{GR}}$ of the proportion θ_i is based only on the sample of the ith domain. The domain sample sizes n_i are small for some domains and there the design variances $\psi_i = \mathrm{var}_{\mathrm{p}}(\hat{\theta}_i^{\mathrm{GR}})$ are large.
- The direct estimators (Rao and Molina, 2015)

$$\hat{\psi}_i^{\mathrm{GR}} = \frac{1}{N_i^2} \sum_{k \in s_i} \sum_{l \in s_i} (1 - \pi_k \pi_l / \pi_{kl}) \frac{(y_k - \mathbf{x}_k' \widehat{\mathbf{B}}_i)(y_l - \mathbf{x}_l' \widehat{\mathbf{B}}_i)}{\pi_k \pi_l}$$

  of ψ_i, where $\pi_{kl} = \mathrm{P}_{\mathrm{p}}\{k, l \in s\} > 0$, have high variances themselves for small samples s_i.
- The true proportions are often very small in the estimation domains. For example, the five-number summary for 16 religions in 60 municipalities (2011 complete data) is

$$(0.000000, 0.000095, 0.000681, 0.007380, 0.922597).$$

Preliminaries for small area estimation

- $(\hat{\theta}_i^{\mathrm{GR}}, \hat{\psi}_i^{\mathrm{GR}})$ are the direct estimators for $i = 1, \ldots, M$.
- $\mathbf{z}_i = (1, z_i)'$ is auxiliary information for the ith domain, where z_i is the proportion of the corresponding variable from the previous complete census 2011.

Using the approximation $\boxed{\psi_i \approx D_i \theta_i (1 - \theta_i)/n_i}$ by Kish (1995) and assuming that the design effects $D_i = c$ for all $i = 1, \ldots, M$,

$$\hat{\psi}_i^{\mathrm{s}} = \hat{c} z_i (1 - z_i)/n_i, \quad \text{where} \quad \hat{c} = \frac{N^2 \hat{\psi}^{\mathrm{s}}}{\sum_{i=1}^{M} \widetilde{N}_i^2 z_i (1 - z_i)/n_i},$$

are smoothed versions of the variances $\hat{\psi}_i^{\mathrm{GR}}$, $i = 1, \ldots, M$.

Here $\hat{\psi}^{\mathrm{s}}$ smooths the direct estimator $\hat{\psi}^{\mathrm{GR}}$ of the variance of the calibrated estimator for the whole population proportion, and $\widetilde{N}_i$ is the size of the ith domain in census 2011.

Synthetic estimation

1. In the case of a *not very small* proportion θ_i, we apply the regression-synthetic estimator

$$\hat{\theta}_i^{\mathrm{S}} = \mathbf{z}_i' \hat{\boldsymbol{\beta}} \quad \text{with} \quad \hat{\boldsymbol{\beta}} = \left(\sum_{i=1}^{M} \frac{\mathbf{z}_i \mathbf{z}_i'}{\hat{\psi}_i} \right)^{-1} \sum_{i=1}^{M} \frac{\mathbf{z}_i \hat{\theta}_i^{\mathrm{GR}}}{\hat{\psi}_i},$$

   which is obtained from the basic domain-level model for EBLUP ignoring random area effects (Rao and Molina, 2015). Here we take

$$\hat{\psi}_i = \hat{\psi}_i^{\mathrm{c}} = \max\{\hat{\psi}_i^{\mathrm{s}}, \hat{\psi}_i^{\mathrm{GR}}\}$$

   according to Čiginas (2022).

2. For a *very small* proportion θ_i, we apply the synthetic estimator

$$\hat{\theta}_i^{\mathrm{S}} = z_i,$$

   which is a constant.

Design-based composite estimation

To estimate the domain proportions θ_i, we apply the composite (shrinkage) estimators (Čiginas, 2022)

$$\hat{\theta}_i^{\mathrm{C}} = \hat{\lambda}_i \hat{\theta}_i^{\mathrm{GR}} + (1 - \hat{\lambda}_i)\hat{\theta}_i^{\mathrm{S}} \quad \text{with} \quad \hat{\lambda}_i = \frac{\min\{\hat{\psi}_i^{\mathrm{s}}, \hat{\psi}_i^{\mathrm{GR}}\}}{\hat{\psi}_i^{\mathrm{c}}}.$$

That estimation is based on the monotonicity of the function $\psi_i \approx \psi(\theta_i) := D_i \theta_i (1 - \theta_i)/n_i$. That is if the direct estimator $\hat{\theta}_i^{\mathrm{GR}}$ is an outlier by its small or large value, then relatively more weight is attached to the synthetic part $\hat{\theta}_i^{\mathrm{S}}$.

Assuming that $\hat{\theta}_i^{\mathrm{C}}$ approximates an optimal linear combination of $\hat{\theta}_i^{\mathrm{GR}}$ and $\hat{\theta}_i^{\mathrm{S}}$ quite well, we apply the estimator (Čiginas, 2021)

$$\mathrm{mse}_{\mathrm{b}}(\hat{\theta}_i^{\mathrm{C}}) = \hat{\lambda}_i (1 - \hat{\lambda}_i) \hat{\psi}_i^{\mathrm{s}} + \hat{\sigma}^2(\hat{\theta}_i^{\mathrm{C}})$$

of $\mathrm{MSE}_{\mathrm{p}}(\hat{\theta}_i^{\mathrm{C}})$, where the term $\hat{\sigma}^2(\hat{\theta}_i^{\mathrm{C}})$ is an estimator of $\mathrm{var}_{\mathrm{p}}(\hat{\theta}_i^{\mathrm{C}})$. We use Rao et al. (1992) bootstrap to estimate the latter variance.

Benchmarking

Let $\hat{\theta}^{\mathrm{C}}_{ij}$, $i = 1, \ldots, M$, be the estimates for $j = 1, \ldots, J$ categories ($J = 16$ for religions). Let $\hat{\theta}^{\mathrm{c}}_{j}$ be the final estimate of the whole population proportion θ_j for the jth category.

We require that

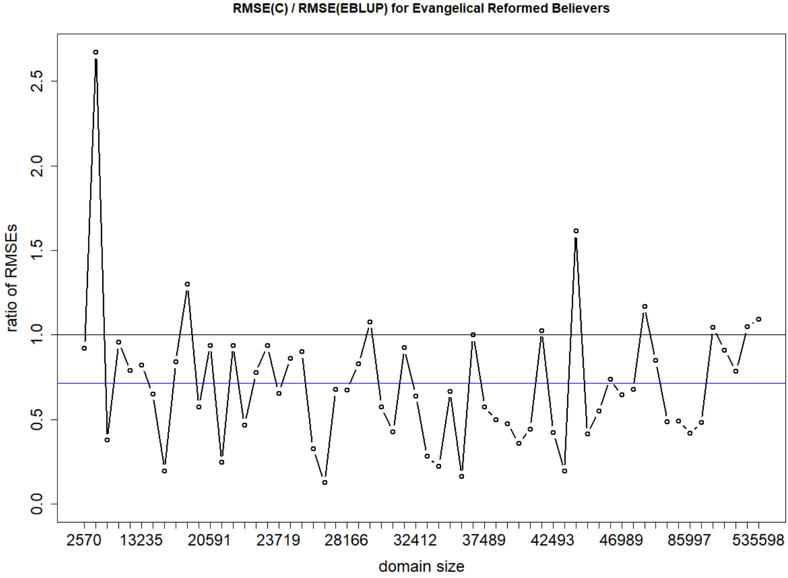
$$\frac{1}{N}\sum_{i=1}^{M} N_i \hat{\theta}^{\mathrm{C}}_{ij} = \hat{\theta}^{\mathrm{c}}_{j} \quad \text{for} \quad j = 1, \ldots, J$$

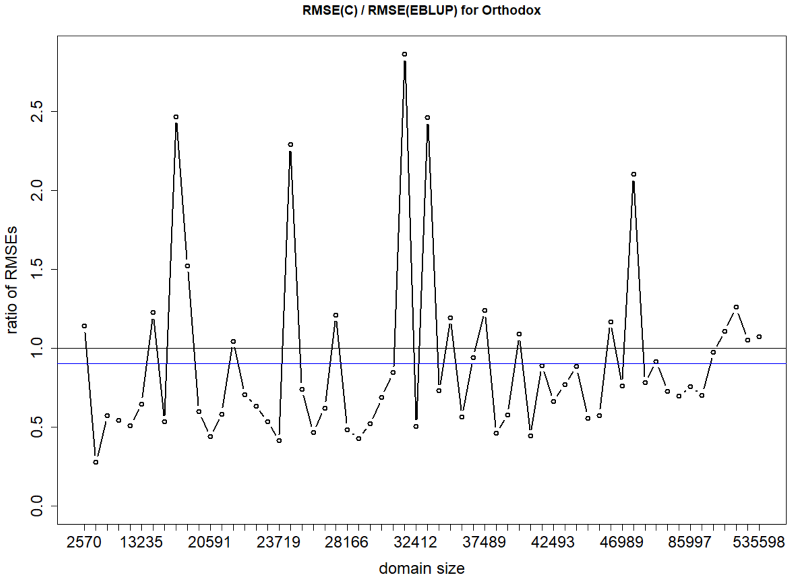
and

$$\sum_{j=1}^{J} \hat{\theta}^{\mathrm{C}}_{ij} = 1 \quad \text{for} \quad i = 1, \ldots, M.$$

The estimates $\hat{\theta}^{\mathrm{C}}_{ij}$, $i = 1, \ldots, M$, $j = 1, \ldots, J$, are benchmarked to satisfy the above conditions using the criterion of weighted least squares with the inverse MSE estimates $\mathrm{mse}_{\mathrm{b}}(\hat{\theta}^{\mathrm{C}}_{i})$ as the weights (Boonstra et al., 2008).

Simulation with 2011 data. Composition vs EBLUP




RMSE(C) / RMSE(EBLUP) for Orthodox
ratio of RMSEs
0.0
0.5
1.0
1.5
2.0
2.5
2570
13235
20591
23719
28166
32412
37489
42493
46989
85997
535598
domain size

Summary

- Due to the very small true proportions, the small area estimation may be relevant for any division of the population.
- Applied design-based composite shrinkage estimation supported by domain-level models is robust compared to unit-level alternatives.
- Domain-level information available from the previous full census is crucial for the efficiency of the estimators. That means more challenges in the next census.

References

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